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Hybrid Equations of Motion for Flexible Multibody Systems Using Quasicoordinates

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A variety of engineering systems, such as automobiles, aircraft, rotorcraft, robots, spacecraft, etc., can be modeled as flexible multibody systems. The individual flexible bodies are in general characterized by distributed parameters. In most earlier investigations they were approximated by some spatial discretization procedure, such as the classical Rayleigh–Ritz method or the finite element method. This paper presents a mathematical formulation for distributed-parameter multibody systems consisting of a set of hybrid (ordinary and partial) differential equations of motion in terms of quasicoordinates. Moreover, the equations for the elastic motions include rotatory inertia and shear deformation effects. The hybrid set is cast in state form, thus making it suitable for control design.

I. Introduction

A PROBLEM of current interest is the dynamics and control of multibody systems. Indeed, a variety of engineering systems, such as automobiles, aircraft, rotorcraft, robots, spacecraft, etc., can be modeled as multibodies. In many engineering applications the bodies can be assumed to be rigid.[1–11] In many other applications, the flexibility effects have to be included.[12–23] For the most part, flexible bodies have distributed mass and stiffness properties, which is likely to cause difficulties in producing a solution. As a result, it is common practice to approximate distributed systems by discrete ones through spatial discretization, which can be carried out by means of the classical Rayleigh–Ritz method or the finite element method.[24] The discretization process amounts to elimination of the spatial coordinates. The equations of motion for the discretized system are derived quite often by the standard Lagrangian approach. For more complex motions, an approach using quasicoordinates seems to offer many advantages.[25–28]

Quite recently, there has been some interest in working with distributed models as much as possible, thus avoiding truncation problems arising from spatial discretization. Consistent with this, hybrid (ordinary and partial) differential equations of motion have been derived for flexible multibody systems[29,30] using the approach of Ref. 24. Hybrid equations of motion in terms of quasicoordinates have been derived for the first time in Ref. 25 for a spinning rigid body with flexible appendages and generalized later[31] for a flexible body undergoing rigid-body and elastic motions. This paper extends the general theory developed[31] to systems of flexible multibodies. In addition, the equations for the elastic motions include rotatory inertia and shear deformation effects.

II. Kinematics

We are concerned with structures consisting of a chain of articulated bodies i ($i = 1, 2, \ldots, N$), which implies that two adjacent bodies $i-1$ and i are hinged at O_i (Fig. 1). To describe the motion of the system, it will prove convenient to conceive of a set of body axes $x_i y_i z_i$ with the origin at O_i and attached to body i in undeformed state. The bodies are assumed to be slender, with axis x_i coinciding with the long axis of the body. As the body deforms, x_i remains tangent to the body at O_i. At the same time, we consider another set of body axes $x_i' y_i' z_i'$, referred to as intermediate axes, with the origin at O_i and attached to body $i-1$ so that x_i' is along the long axis. We will also find it convenient to introduce an inertial frame of reference XYZ with the origin at O.

We denote the position vector of point O_i relative to the origin O by $\boldsymbol{R}_{oi} = [X_{oi}\ Y_{oi}\ Z_{oi}]^T$. Then, we denote the position of a typical point $\mathcal{P}_i$ in the undeformed i body relative to O_i by $\boldsymbol{r}_i$ and the elastic displacement of $\mathcal{P}_i$ by $\boldsymbol{u}_i$. Hence, the radius vector from O to $\mathcal{P}_i$ in displaced position is simply

$$\boldsymbol{R}_i = C_i^* \boldsymbol{R}_{oi} + \boldsymbol{r}_i + \boldsymbol{u}_i \qquad i = 1, 2, \ldots, N \tag{1}$$

where C_i^* is the matrix of direction cosines of axes $x_i y_i z_i$ with respect to axes $x_{i-1} y_{i-1} z_{i-1}$, and note that the vector $\boldsymbol{R}_{oi}$ is in terms of components along the body axes $x_{i-1} y_{i-1} z_{i-1}$ and the vectors $\boldsymbol{R}_i$, $\boldsymbol{r}_i$, and $\boldsymbol{u}_i$ are in terms of components along the body axes $x_i y_i z_i$.

We consider here bodies in the form of bars with the long axis x_i passing through O_i and O_{i+1} when the bars are undeformed. We are concerned with bars undergoing torsion about axis x_i and bending about axes y_i and z_i as well as shearing distortion in the y_i and z_i directions. Then, the vectors $\boldsymbol{r}_i$ and $\boldsymbol{u}_i$ can be written in the more explicit form

$$\boldsymbol{r}_i = [x_i \quad 0 \quad 0]^T \tag{2a}$$

$$\boldsymbol{u}_i(x_i, t) = [0 \quad u_{yi}(x_i, t) \quad u_{zi}(x_i, t)]^T \tag{2b}$$

The radius vector $\boldsymbol{R}_{oi}$ depends on the motion of the preceding $i-1$ bodies in the chain. In particular, we can write the following recursive relation:

$$\boldsymbol{R}_{oi} = C_{i-1}^* \boldsymbol{R}_{o,i-1} + \boldsymbol{r}_{i-1}(l_{i-1}) + \boldsymbol{u}_{i-1}(l_{i-1}, t) \quad i = 2, 3, \ldots, N \tag{3}$$

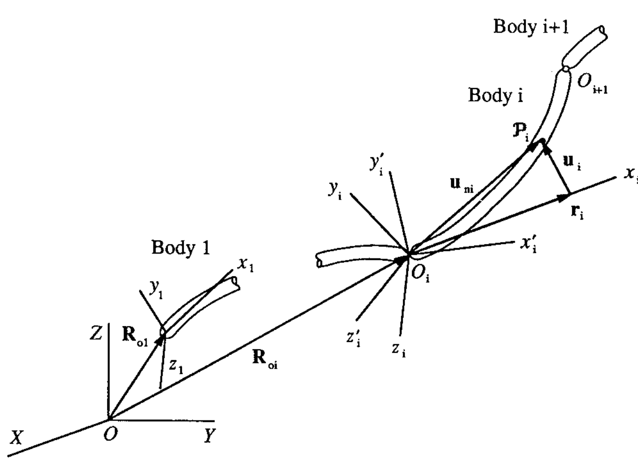

Fig. 1 Flexible multibody structure.

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where l_{i-1} is the length of body $i-1$. Note that $\boldsymbol{R}_{o1} = \boldsymbol{R}_{o1}(t)$ is simply the radius vector from O to the origin O_1 of the body axes of the first body in the chain.

At this point, we propose to define the rotational motions. In the first place, it will prove convenient to introduce a set of body axes $\xi_i\eta_i\zeta_i$ attached to a typical beam cross section originally in the nominal position $x_iy_iz_i$ and moving with the cross section as body i deforms. In this regard, note that $\xi_{i-1}(l_{i-1})\eta_{i-1}(l_{i-1})\zeta_{i-1}(l_{i-1})$ coincide with $x_i'y_i'z_i'$. Then, denoting the angle of twist by ψ_{xi} and the bending rotation angles by ψ_{yi} and ψ_{zi}, we conclude that axes $\xi_i\eta_i\zeta_i$ experience the angular displacement

$$\boldsymbol{\psi}_i(x_i, t) = [\psi_{xi}(x_i, t) \quad \psi_{yi}(x_i, t) \quad \psi_{zi}(x_i, t)]^T \tag{4}$$

with respect to axes $x_iy_iz_i$. On various occasions throughout this paper, we encounter skew symmetric matrices derived from vectors. As an example, if a typical vector $\boldsymbol{r}$ has components x, y, and z, then the associated skew symmetric matrix has the form

$$\tilde{r} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{5}$$

In view of this definition, the matrix of direction cosines of $\xi_i\eta_i\zeta_i$ relative to $x_iy_iz_i$ can be shown to have the expression

$$E_i(x_i, t) = I - \tilde{\psi}_i(x_i, t) \tag{6}$$

in which I is the 3×3 identity matrix, and we note that Eq. (6) follows from the assumption that the components of ψ_i are small. Next, we assume that axes $x_iy_iz_i$ are obtained from axes $x_i'y_i'z_i'$ through the rotations θ_{ij}, where j can take the values 1 or 1, 2 or 1, 2, 3, depending on the nature of the hinge at O_i and denote by $C_i(\boldsymbol{\theta}_i)$ the matrix of direction cosines of $x_iy_iz_i$ relative to $x_i'y_i'z_i'$, where $\boldsymbol{\theta}_i = [\theta_{i1}\ \theta_{i2}\ \theta_{i3}]^T$. Then, the matrix of direction cosines of axes $x_iy_iz_i$ relative to axes $x_{i-1}y_{i-1}z_{i-1}$ is simply

$$C_i^* = C_iE_{i-1}(l_{i-1}, t) \tag{7}$$

From kinematics, the velocity vector of the typical point $\mathcal{P}_i$ in displaced position in terms of the rotating body axes $x_iy_iz_i$, has the expression

$$\begin{aligned} \boldsymbol{V}_i &= \boldsymbol{V}_{oi} + \tilde{\boldsymbol{\Omega}}_{ri}(\boldsymbol{r}_i + \boldsymbol{u}_i) + \boldsymbol{v}_i \\ &= \boldsymbol{V}_{oi} + (\tilde{r}_i + \tilde{u}_i)^T\boldsymbol{\Omega}_{ri} + \boldsymbol{v}_i \qquad i = 1, 2, \ldots, N \end{aligned} \tag{8}$$

where $\boldsymbol{V}_{oi}$ is the velocity vector of the origin O_i, $\boldsymbol{\Omega}_{ri}$ is the angular velocity vector of axes $x_iy_iz_i$ relative to axes XYZ, and

$$\boldsymbol{v}_i(x_i, t) = \dot{\boldsymbol{u}}_i(x_i, t) \tag{9}$$

is the elastic velocity vector relative to $x_iy_iz_i$, all in terms of $x_iy_iz_i$ components. We note that the velocity vector of point O_i can be written in the recursive form

$$\begin{aligned} \boldsymbol{V}_{oi} &= C_i^*\boldsymbol{V}_{i-1}(l_{i-1}, t) \\ &= C_i^*\big\{\boldsymbol{V}_{o,i-1} + [\tilde{r}_{i-1}(l_{i-1}) + \tilde{u}_{i-1}(l_{i-1}, t)]^T\boldsymbol{\Omega}_{r,i-1} \\ &\quad + \boldsymbol{v}_{i-1}(l_{i-1}, t)\big\} \qquad i = 2, 3, \ldots, N \end{aligned} \tag{10}$$

Moreover, introducing the notation

$$\boldsymbol{\Omega}_{ei}(x_i, t) = \dot{\boldsymbol{\psi}}_i(x_i, t) \qquad i = 1, 2, \ldots, N \tag{11}$$

the angular velocity vector of the cross-sectional axes $\xi_i\eta_i\zeta_i$ relative to the inertial space is simply

$$\boldsymbol{\Omega}_i = \boldsymbol{\Omega}_{ri} + \boldsymbol{\Omega}_{ei}(x_i, t) \qquad i = 1, 2, \ldots, N \tag{12}$$

Finally, letting $\boldsymbol{\omega}_i$ be the angular velocity vector of axes $x_iy_iz_i$ relative to axes $x_i'y_i'z_i'$, in terms of $x_iy_iz_i$ components, the angular velocity vector of $x_iy_iz_i$ is given by the recursive formula

$$\begin{aligned} \boldsymbol{\Omega}_{ri} &= C_i^*\boldsymbol{\Omega}_{i-1}(l_{i-1}, t) + \boldsymbol{\omega}_i \\ &= C_i^*[\boldsymbol{\Omega}_{r,i-1} + \boldsymbol{\Omega}_{e,i-1}(l_{i-1}, t)] + \boldsymbol{\omega}_i \qquad i = 2, 3, \ldots, N \end{aligned} \tag{13}$$

where the second equality follows from Eq. (12).

III. Standard Lagrange's Equations for Flexible Multibody Systems

The motion of our multibody system is described in terms of rigid-body displacements of sets of body axes and elastic displacements relative to these body axes. As a result, the equations of motion are hybrid, in the sense that they consist of ordinary differential equations for the rigid-body displacements and partial differential equations for the elastic displacements. The equations of motion can be derived by means of the extended Hamilton principle,[32] which can be stated in the form

$$\int_{t_1}^{t_2}(\delta L + \overline{\delta W})\,\mathrm{d}t = 0, \qquad \delta\boldsymbol{q} = 0, \qquad \delta\boldsymbol{u}_i = \delta\boldsymbol{\psi}_i = 0$$
$$i = 1, 2, \ldots, N \quad \text{at} \quad t = t_1, t_2 \tag{14}$$

where

$$L = T - V \tag{15}$$

is the Lagrangian, in which T is the kinetic energy and V is the potential energy, and $\overline{\delta W}$ is the virtual work. Moreover, $\boldsymbol{q}$ is the rigid-body displacement vector, and $\boldsymbol{u}_i$, $\boldsymbol{\psi}_i$ $(i = 1, 2, \ldots, N)$ are the elastic displacement vectors introduced earlier. Hence, before we can derive equations of motion, we must derive general expressions for T, V, and $\overline{\delta W}$.

Taking the x_i axis to coincide with the centroidal axis of the undeformed beam, the kinetic energy can be shown to consist of two parts, one due to translations and one due to rotations.[24] Hence, using Eqs. (8) and (12), the kinetic energy can be expressed in the form

$$T = \sum_{i=1}^{N}\int_0^{l_i}\hat{T}_i\,\mathrm{d}x_i \tag{16}$$

where

$$\begin{aligned} \hat{T}_i &= \tfrac{1}{2}\big(\rho_i\boldsymbol{V}_i^T\boldsymbol{V}_i + \boldsymbol{\Omega}_i^T\hat{J}_{ci}\boldsymbol{\Omega}_i\big) \\ &= \tfrac{1}{2}\big(\rho_i\boldsymbol{V}_{oi}^T\boldsymbol{V}_{oi} + \boldsymbol{\Omega}_{ri}^T\hat{J}_i\boldsymbol{\Omega}_{ri} + \rho_i\dot{\boldsymbol{u}}_i^T\dot{\boldsymbol{u}}_i + 2\boldsymbol{V}_{oi}^T\tilde{\hat{S}}_i^T\boldsymbol{\Omega}_{ri} + 2\rho_i\boldsymbol{V}_{oi}^T\dot{\boldsymbol{u}}_i \\ &\quad + 2\boldsymbol{\Omega}_{ri}^T\tilde{\hat{S}}_i\dot{\boldsymbol{u}}_i + \boldsymbol{\Omega}_{ri}^T\hat{J}_{ci}\boldsymbol{\Omega}_{ri} + \dot{\boldsymbol{\psi}}_i^T\hat{J}_{ci}\dot{\boldsymbol{\psi}}_i + 2\boldsymbol{\Omega}_{ri}^T\hat{J}_{ci}\dot{\boldsymbol{\psi}}_i\big) \\ &= \tfrac{1}{2}\big[\rho_i\boldsymbol{V}_{oi}^T\boldsymbol{V}_{oi} + \boldsymbol{\Omega}_{ri}^T\hat{J}_{ti}\boldsymbol{\Omega}_{ri} + \rho_i\dot{\boldsymbol{u}}_i^T\dot{\boldsymbol{u}}_i + \dot{\boldsymbol{\psi}}_i^T\hat{J}_{ci}\dot{\boldsymbol{\psi}}_i + 2\boldsymbol{V}_{oi}^T\tilde{\hat{S}}_i^T\boldsymbol{\Omega}_{ri} \\ &\quad + 2\rho_i\boldsymbol{V}_{oi}^T\dot{\boldsymbol{u}}_i + 2\boldsymbol{\Omega}_{ri}^T(\tilde{\hat{S}}_i\dot{\boldsymbol{u}}_i + \hat{J}_{ci}\dot{\boldsymbol{\psi}}_i)\big] \end{aligned} \tag{17}$$

is the kinetic energy density of member i, in which ρ_i is the mass density and

$$\hat{J}_{ti} = \hat{J}_i + \hat{J}_{ci} \tag{18}$$

is the total moment of inertia density matrix, where

$$\begin{aligned} \hat{J}_i &= \rho_i(\tilde{r}_i + \tilde{u}_i)(\tilde{r}_i + \tilde{u}_i)^T \\ &= \rho_i\begin{bmatrix} u_{yi}^2 + u_{zi}^2 & -x_iu_{yi} & -x_iu_{zi} \\ -x_iu_{yi} & x_i^2 + u_{zi}^2 & -u_{yi}u_{zi} \\ -x_iu_{zi} & -u_{yi}u_{zi} & x_i^2 + u_{yi}^2 \end{bmatrix} \end{aligned} \tag{19a}$$

and

$$\hat{J}_{ci} = \mathrm{diag}[\hat{J}_{xixi} \quad \hat{J}_{yiyi} \quad \hat{J}_{zizi}] \tag{19b}$$

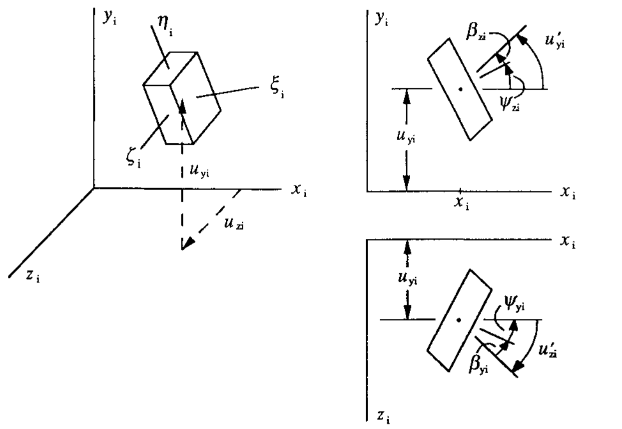

Fig. 2 Rotations and shear deformations of a typical differential element.

in which $\hat{J}_{xixi}$, $\hat{J}_{yiyi}$, and $\hat{J}_{zizi}$ are cross-sectional mass moments of inertia densities, and note that, because the elastic deformations are relatively small, they are approximately equal to $\hat{J}_{\xi i\xi i}$, $\hat{J}_{\eta i\eta i}$ and $\hat{J}_{\zeta i\zeta i}$, respectively. Moreover, $\hat{S}_i$ is obtained from

$$\hat{S}_i = \rho_i(\boldsymbol{r}_i + \boldsymbol{u}_i) = \rho_i[x_i \quad u_{yi} \quad u_{zi}]^T \tag{20}$$

which is recognized as the first moments of inertia density vector.

Assuming that differential gravity effects are negligibly small, the potential energy reduces to the strain energy. As indicated earlier, the elastic members undergo torsion about x_i and bending about y_i and z_i as well as shearing distortions in the y_i and z_i directions. Referring to Fig. 2, we conclude that the relations between the bending displacements u_{yi} and u_{zi}, the bending angular displacements ψ_{yi} and ψ_{zi}, and the shearing distortion angles β_{yi} and β_{zi} are

$$u'_{yi} = \psi_{zi} + \beta_{zi} \tag{21a}$$

$$u'_{zi} = -\psi_{yi} - \beta_{yi} \tag{21b}$$

where primes denote partial derivatives with respect to x_i. From mechanics of materials, the relation between the twisting moment M_{xi} and the twist angle ψ_{xi} is simply

$$M_{xi} = k_{xi} G_i I_{xi} \psi'_{xi} \tag{22}$$

where k_{xi} is a factor depending on the shape of the cross section and $G_i I_{xi}$ is the torsional rigidity, in which G_i is the shear modulus and I_{xi} is the polar area moment of inertia about axis x_i. Moreover, the bending moments are related to the bending rotational displacements by

$$M_{yi} = E_i I_{yi} \psi'_{yi} \tag{23a}$$

$$M_{zi} = E_i I_{zi} \psi'_{zi} \tag{23b}$$

in which E_i is Young's modulus and I_{yi} and I_{zi} are area moments of inertia about axes parallel to y_i and z_i, respectively, and passing through the center of the cross-sectional area, and the shearing forces are related to the shearing distortion angles according to

$$Q_{yi} = k_{zi} G_i A_i \beta_{zi} \tag{24a}$$

$$Q_{zi} = -k_{yi} G_i A_i \beta_{yi} \tag{24b}$$

where k_{yi} and k_{zi} are factors depending on the shape of the cross-sectional area, G_i is the shear modulus, and A_i is the cross-sectional area.

The strain energy can be expressed as

$$V = \sum_{i=1}^{N} \int_0^{l_i} \hat{V}_i \, \mathrm{d}x_i \tag{25}$$

where, using Eqs. (21–24),

$$\begin{aligned} \hat{V}_i &= \tfrac{1}{2}(M_{xi}\psi'_{xi} + M_{yi}\psi'_{yi} + M_{zi}\psi'_{zi} + Q_{yi}\beta_{zi} - Q_{zi}\beta_{yi}) \\ &= \tfrac{1}{2}\big[k_{xi}G_i I_{xi}(\psi'_{xi})^2 + E_i I_{yi}(\psi'_{yi})^2 + E_i I_{zi}(\psi'_{zi})^2 \\ &\quad + k_{yi}G_i A_i(u'_{yi} - \psi_{zi})^2 + k_{zi}G_i A_i(u'_{zi} + \psi_{yi})^2\big] \end{aligned} \tag{26}$$

is the potential energy density for member i.

Next, we wish to develop an expression for the virtual work due to nonconservative actuator forces and torques. Using the analogy with Eqs. (8) and (12), the virtual work can be written in the form

$$\begin{aligned} \overline{\delta W} &= \sum_{i=1}^{N}\left[\int_0^{l_i}\left(\boldsymbol{f}_i^T\delta\boldsymbol{R}_i^* + \boldsymbol{m}_i^T\delta\boldsymbol{\Theta}_i^*\right)\mathrm{d}x_i\right] + \sum_{i=2}^{N}\boldsymbol{M}_{oi}^{*T}\delta\boldsymbol{\theta}_i^* \\ &= \sum_{i=1}^{N}\left\{\int_0^{l_i}\left[\boldsymbol{f}_i^T\left(\delta\boldsymbol{R}_{oi}^* + \tilde{\boldsymbol{r}}_i^T\delta\boldsymbol{\Theta}_{ri}^* + \delta\boldsymbol{u}_i\right)\right.\right. \\ &\quad \left.\left. + \boldsymbol{m}_i^T(\delta\boldsymbol{\Theta}_{ri}^* + \delta\boldsymbol{\psi}_i)\right]\mathrm{d}x_i\right\} + \sum_{i=2}^{N}\boldsymbol{M}_{oi}^{*T}\delta\boldsymbol{\theta}_i^* \\ &= \sum_{i=1}^{N}\left[\boldsymbol{F}_{ri}^{*T}\delta\boldsymbol{R}_{oi}^* + \boldsymbol{M}_{ri}^{*T}\delta\boldsymbol{\Theta}_{ri}^*\right. \\ &\quad \left. + \int_0^{l_i}\left(\boldsymbol{f}_i^T\delta\boldsymbol{u}_i + \boldsymbol{m}_i^T\delta\boldsymbol{\psi}_i\right)\mathrm{d}x_i\right] + \sum_{i=2}^{N}\boldsymbol{M}_{oi}^{*T}\delta\boldsymbol{\theta}_i^* \end{aligned} \tag{27}$$

in which $\boldsymbol{f}_i$ and $\boldsymbol{m}_i$ are distributed actuator forces and torques acting over the domain i, $\boldsymbol{M}_{oi}^*$ are torque actuators located at points O_i and acting on both members $i-1$ and i, for $i = 2, 3, \ldots, N$, $\delta\boldsymbol{R}_i^*$ is the virtual displacement vector of point $\mathcal{P}_i$, $\delta\boldsymbol{\Theta}_i^*$ is the virtual rotation vector of axes $\xi_i\eta_i\zeta_i$, $\delta\boldsymbol{\theta}_i^*$ is the virtual rotation vector of axes $x_i y_i z_i$ relative to axes $x'_i y'_i z'_i$, $\delta\boldsymbol{R}_{oi}^*$ is the virtual displacement vector of point O_i, and $\delta\boldsymbol{\Theta}_{ri}^*$ is the virtual rotation vector of axes $x_i y_i z_i$ relative to axes XYZ, where all of these vectors are in terms of components along axes $x_i y_i z_i$, and asterisks indicate quasicoordinates[32] and associated forces and torques. Note that the term $\boldsymbol{f}_i^T\tilde{\boldsymbol{u}}_i^T\delta\boldsymbol{\Theta}_{ri}^*$ was omitted from $\delta\boldsymbol{R}_i^*$ on the basis that it is second order in magnitude. Moreover,

$$\boldsymbol{F}_{ri}^* = \int_0^{l_i}\boldsymbol{f}_i\,\mathrm{d}x_i \tag{28a}$$

$$\boldsymbol{M}_{ri}^* = \int_0^{l_i}(\tilde{\boldsymbol{r}}_i\boldsymbol{f}_i + \boldsymbol{m}_i)\,\mathrm{d}x_i \tag{28b}$$

are, respectively, resultant forces and torques acting on member i.

Before proceeding with the derivation of Lagrange's equations by means of the extended Hamilton principle, Eq. (14), it is advisable to identify a set of generalized coordinates capable of describing the motion of the system fully. From Eq. (3), we conclude that the motion of only one of the points O_i is independent. We choose this point as O_1, so that we retain only $\boldsymbol{R}_{o1}(t)$ for inclusion in the set of generalized coordinates. On the other hand, because O_i represent hinge points, the rigid-body rotation vectors $\boldsymbol{\theta}_i(t)$ $(i = 1, 2, \ldots, N)$ are all independent. Similarly, the nonzero components of the elastic displacement and rotation vectors $\boldsymbol{u}_i(x_i, t)$ and $\boldsymbol{\psi}_i(x_i, t)$ $(i = 1, 2, \ldots, N)$, respectively, are also all independent. It will prove convenient to introduce the rigid-body motion vector

$$\boldsymbol{q}(t) = \left[\boldsymbol{R}_{o1}^T(t) \quad \boldsymbol{\theta}_1^T(t) \quad \boldsymbol{\theta}_2^T(t) \quad \cdots \quad \boldsymbol{\theta}_N^T(t)\right]^T \tag{29}$$

so that we propose to derive a vector Lagrange ordinary differential equation for $\boldsymbol{q}(t)$ and N pairs of vector Lagrange partial differential equations for $\boldsymbol{u}_i(x_i, t)$ and $\boldsymbol{\psi}_i(x_i, t)$ $(i = 1, 2, \ldots, N)$. To this end, we wish to express the Lagrangian in general functional form, and we note that the Lagrangian contains not only $\boldsymbol{q}$, $\boldsymbol{u}_i$, and $\boldsymbol{\psi}_i$ but also time and spatial derivatives of these vectors. Moreover, we observe from Eqs. (3), (7), (10), and (13) that the Lagrangian contains terms involving $\boldsymbol{u}_i(l_i,t), \dot{\boldsymbol{u}}_i(l_i,t), \boldsymbol{\psi}_i(l_i,t)$, and $\dot{\boldsymbol{\psi}}_i(l_i,t)$. Such terms will contribute to the dynamic boundary conditions accompanying the partial differential equations for $\boldsymbol{u}_i(x_i,t)$ and $\boldsymbol{\psi}_i(x_i,t)$. In view of this, we express the Lagrangian in the general form

$$L = L[\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}_i, \boldsymbol{u}_i', \dot{\boldsymbol{u}}_i, \boldsymbol{\psi}_i, \boldsymbol{\psi}_i', \dot{\boldsymbol{\psi}}_i, \boldsymbol{u}_i(l_i,t) \quad \dot{\boldsymbol{u}}_i(l_i,t), \boldsymbol{\psi}_i(l_i,t), \dot{\boldsymbol{\psi}}_i(l_i,t)] \tag{30}$$

The extended Hamilton principle, Eq. (14), calls for the variation of the Lagrangian, which can be expressed symbolically as

$$\begin{aligned}\delta L = &\left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^T \delta\boldsymbol{q} + \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^T \delta\dot{\boldsymbol{q}} \\ &+ \sum_{i=1}^{N}\int_0^{l_i}\left[\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}_i}\right)^T \delta\boldsymbol{u}_i + \left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\right)^T \delta\boldsymbol{u}'_i + \cdots \right. \\ &\left. + \left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right)^T \delta\dot{\boldsymbol{\psi}}_i\right]\mathrm{d}x_i + \sum_{i=1}^{N}\left\{\left[\frac{\partial L}{\partial \boldsymbol{u}_i(l_i,t)}\right]^T \delta\boldsymbol{u}_i(l_i,t)\right. \\ &+ \left[\frac{\partial L}{\partial \dot{\boldsymbol{u}}_i(l_i,t)}\right]^T \delta\dot{\boldsymbol{u}}_i(l_i,t) + \left[\frac{\partial L}{\partial \boldsymbol{\psi}_i(l_i,t)}\right]^T \delta\boldsymbol{\psi}_i(l_i,t) \\ &\left. + \left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(l_i,t)}\right]^T \delta\dot{\boldsymbol{\psi}}_i(l_i,t)\right\}\end{aligned} \tag{31}$$

where $\hat{L}_i = \hat{T}_i - \hat{V}_i$ is the Lagrangian density for body i. Moreover, $(\partial L/\partial \boldsymbol{q})^T$ represents the row matrix $[\partial L/\partial q_1\ \partial L/\partial q_2 \cdots \partial L/\partial q_{N_R}]$, etc., where N_R is the total number of independent rigid-body degrees of freedom. Consistent with the generalized coordinates used, the virtual work has the form

$$\begin{aligned}\overline{\delta W} = &\boldsymbol{Q}^T\delta\boldsymbol{q} + \sum_{i=1}^{N}\int_0^{l_i}\left(\boldsymbol{f}_i^T\delta\boldsymbol{u}_i + \boldsymbol{m}_i^T\delta\boldsymbol{\psi}_i\right)\mathrm{d}x_i \\ &+ \sum_{i=1}^{N}\left[\boldsymbol{U}_i^T\delta\boldsymbol{u}_i(l_i,t) + \boldsymbol{\Psi}_i^T\delta\boldsymbol{\psi}_i(l_i,t)\right]\end{aligned} \tag{32a}$$

where we write the generalized force vector $\boldsymbol{Q}$ in the form

$$\boldsymbol{Q} = \left[\boldsymbol{F}_1^T \quad \boldsymbol{M}_1^T \quad \boldsymbol{M}_2^T \quad \cdots \quad \boldsymbol{M}_N^T\right]^T \tag{32b}$$

and note that $\boldsymbol{F}_1$ is a generalized force and $\boldsymbol{M}_1,\ldots,\boldsymbol{M}_N$ are generalized torques. They can all be related to the actuator forces and moments, but we postpone further discussion of this subject and the derivation of specific formulas for $\boldsymbol{U}_i$ and $\boldsymbol{\Psi}_i$ until later.

Introducing Eqs. (31) and (32) into Eq. (14), carrying out the usual integrations by parts, and recalling that the virtual displacements vanish at $t = t_1, t_2$, we have

$$\begin{aligned}\int_{t_1}^{t_2}\Bigg(&\left[\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) + \boldsymbol{Q}\right]^T\delta\boldsymbol{q} \\ &+ \sum_{i=1}^{N}\left\langle\int_0^{l_i}\left\{\left[\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{u}}_i}\right) + \boldsymbol{f}_i\right]^T\delta\boldsymbol{u}_i\right.\right. \\ &\left. + \left[\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right) + \boldsymbol{m}_i\right]^T\delta\boldsymbol{\psi}_i\right\}\mathrm{d}x_i \\ &\left. + \left[\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\right)^T\delta\boldsymbol{u}_i + \left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\right)^T\delta\boldsymbol{\psi}_i\right]\Bigg|_0^{l_i}\right\rangle + \sum_{i=1}^{N-1}\left\langle\left\{\frac{\partial L}{\partial \boldsymbol{u}_i(l_i,t)}\right.\right. \\ &\left. - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{u}}_i(l_i,t)}\right] + \boldsymbol{U}_i\right\}^T\delta\boldsymbol{u}_i(l_i,t) + \left\{\frac{\partial L}{\partial \boldsymbol{\psi}_i(l_i,t)}\right. \\ &\left.\left. - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(l_i,t)}\right] + \boldsymbol{\Psi}_i\right\}^T\delta\boldsymbol{\psi}_i(l_i,t)\right\rangle\Bigg)\mathrm{d}t = 0\end{aligned} \tag{33}$$

Then, invoking the arbitrariness of the virtual displacements, we obtain the system Lagrange equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{Q} \tag{34a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{u}}_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\right) - \frac{\partial \hat{L}_i}{\partial \boldsymbol{u}_i} = \boldsymbol{f}_i \qquad i = 1,2,\ldots,N, \quad 0 < x_i < l_i \tag{34b}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\right) - \frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} = \boldsymbol{m}_i \qquad i = 1,2,\ldots,N, \quad 0 < x_i < l_i \tag{34c}$$

where $\boldsymbol{u}_i$ and $\boldsymbol{\psi}_i$ must be such that the equations

$$\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\right)^T\delta\boldsymbol{u}_i\Bigg|_{x_i=0} = 0 \qquad i = 1,2,\ldots,N \tag{35a}$$

$$\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\right)^T\delta\boldsymbol{\psi}_i\Bigg|_{x_i=0} = 0 \qquad i = 1,2,\ldots,N \tag{35b}$$

$$\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\Bigg|_{x_i=l_i} + \boldsymbol{U}_i - \left\{\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{u}}_i(l_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{u}_i(l_i,t)}\right\}\right)^T \times \delta\boldsymbol{u}_i(l_i,t) = 0 \qquad i = 1,2,\ldots,N-1 \tag{35c}$$

$$\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\Bigg|_{x_i=l_i} + \boldsymbol{\Psi}_i - \left\{\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(l_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{\psi}_i(l_i,t)}\right\}\right)^T \times \delta\boldsymbol{\psi}_i(l_i,t) = 0, \qquad i = 1,2,\ldots,N-1 \tag{35d}$$

$$\frac{\partial \hat{L}_N}{\partial \boldsymbol{u}'_N}\delta\boldsymbol{u}_N(x_N,t)\Bigg|_{x_N=l_N} = 0 \tag{35e}$$

$$\frac{\partial \hat{L}_N}{\partial \boldsymbol{\psi}'_N}\delta\boldsymbol{\psi}_N(x_N,t)\Bigg|_{x_N=l_N} = 0 \tag{35f}$$

must be satisfied. Recalling that the body axes $x_iy_iz_i$ are embedded in the body at $x_i = 0$, we conclude that satisfaction of Eqs. (35) is guaranteed if

$$\boldsymbol{u}_i(0,t) = \boldsymbol{0} \qquad i = 1,2,\ldots,N \tag{36a}$$

$$\boldsymbol{\psi}_i(0,t) = \boldsymbol{0} \qquad i = 1,2,\ldots,N \tag{36b}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{u}}_i(l_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{u}_i(l_i,t)} = \frac{\partial \hat{L}_i}{\partial \boldsymbol{u}'_i}\Bigg|_{x_i=l_i} + \boldsymbol{U}_i \qquad i = 1,2,\ldots,N-1 \tag{36c}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(l_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{\psi}_i(l_i,t)} = \frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\Bigg|_{x_i=l_i} + \boldsymbol{\Psi}_i \qquad i = 1,2,\ldots,N-1 \tag{36d}$$

$$\frac{\partial \hat{L}_N}{\partial \boldsymbol{u}'_N}\Bigg|_{x_N=l_N} = \boldsymbol{0} \tag{36e}$$

$$\frac{\partial \hat{L}_N}{\partial \boldsymbol{\psi}'_N}\Bigg|_{x_N=l_N} = \boldsymbol{0} \tag{36f}$$

Equations (34a) represent ordinary differential equations for the rigid-body motion and Eqs. (34b) and (34c) represent partial differential equations for the elastic motions. Moreover, Eqs. (36)

are recognized as the boundary conditions accompanying the partial differential equations. Although Eqs. (34a), (34b), (36a), (36c), and (36e) on the one hand and Eqs. (34c), (36b), (36d), and (36f) on the other hand have the appearance of being independent sets of equations, they are in fact simultaneous. They constitute a hybrid (ordinary and partial) set of differential equations governing the motion of the multibody system shown in Fig. 1.

IV. Lagrange's Equations for Flexible Multibody Systems in Terms of Quasicoordinates

Equations (34) seem very simple, but they are not. The reason for this is that the kinetic energy is only an implicit function of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ and not an explicit one. The kinetic energy is an explicit function of $\boldsymbol{V}_{oi}$ and $\boldsymbol{\omega}_i$, which are commonly known as derivatives of quasicoordinates.[32] Actually, the kinetic energy is an explicit function of $\boldsymbol{\Omega}_i$, but $\boldsymbol{\Omega}_i$ is related directly to $\boldsymbol{\omega}_i$, as can be seen from Eq. (13). As shown in Ref. 31 for a single flexible body, hybrid Lagrange equations of motion in terms of quasicoordinates are considerably simpler than the standard Lagrange equations. We propose to show in this paper that the same is true for multibodies.

Recalling definition (29) of the rigid-body displacement vector $\boldsymbol{q}(t)$, we can rewrite Eq. (34a) in the more detailed form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{R}}_{o1}}\right) - \frac{\partial L}{\partial \boldsymbol{R}_{o1}} = \boldsymbol{F}_1 \tag{37a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}_i}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}_i} = \boldsymbol{M}_i \qquad i = 1, 2, \ldots, N \tag{37b}$$

The vectors $\boldsymbol{R}_{o1}, \dot{\boldsymbol{R}}_{o1}$ and $\boldsymbol{F}_1$ are in terms of components along the inertial axes XYZ. Moreover, the components of the symbolic vector $\boldsymbol{\theta}_i$ represent rotations about nonorthogonal axes leading from $x_i'y_i'z_i'$ to $x_iy_iz_i$ and the components of $\boldsymbol{M}_i$ are associated moments. An example of such rotations are Euler's angles.[32] As the quasi-velocity counterpart of the generalized velocity vector $\dot{\boldsymbol{q}}(t)$, we choose

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{V}_{o1}^T & \boldsymbol{\omega}_1^T & \boldsymbol{\omega}_2^T & \cdots & \boldsymbol{\omega}_N^T \end{bmatrix}^T \tag{38}$$

and we note that $\boldsymbol{w}$ does not equal the time derivative $\dot{\boldsymbol{q}}$ of the displacements. We also note that every three-dimensional vector entering into $\boldsymbol{w}$ is in terms of the corresponding orthogonal body axes $x_iy_iz_i$. The relation between the velocity vector $\boldsymbol{V}_{o1}$ in terms of body axes and the velocity vector $\dot{\boldsymbol{R}}_{o1}$ in terms of inertial axes is simply

$$\boldsymbol{V}_{o1} = C_1\dot{\boldsymbol{R}}_{o1} \tag{39}$$

where C_1 is the matrix of direction cosines first introduced in Sec. II, and that between the velocity vector $\boldsymbol{\omega}_i$ in terms of body axes and the Eulerian-type velocities $\dot{\boldsymbol{\theta}}_i$ can be written as

$$\boldsymbol{\omega}_i = D_i\dot{\boldsymbol{\theta}}_i \qquad i = 1, 2, \ldots, N \tag{40}$$

where D_i is a given transformation matrix (Ref. 32). Equations (39) and (40) and their reciprocal relations can be expressed in the compact form

$$\boldsymbol{w} = A^T(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{41a}$$

$$\dot{\boldsymbol{q}} = B(\boldsymbol{q})\boldsymbol{w} \tag{41b}$$

where

$$A = \text{block-diag}\begin{bmatrix} C_1^T & D_1^T & D_2^T & \cdots & D_N^T \end{bmatrix} \tag{42a}$$

$$B = \text{block-diag}\begin{bmatrix} C_1^T & D_1^{-1} & D_2^{-1} & \cdots & D_N^{-1} \end{bmatrix} \tag{42b}$$

Equations (37) postulate a Lagrangian in terms of generalized coordinates and velocities, Eq. (30), when in fact the Lagrangian defined by Eqs. (15–17), (25), and (26) is in terms of generalized coordinates and quasi-velocities. To distinguish between the two forms, we define

$$L^* = L^*[\boldsymbol{q}, \boldsymbol{w}, \boldsymbol{u}_i, \boldsymbol{u}_i', \dot{\boldsymbol{u}}_i, \boldsymbol{\psi}_i, \boldsymbol{\psi}_i', \dot{\boldsymbol{\psi}}_i, \ \boldsymbol{u}_i(l_i, t), \dot{\boldsymbol{u}}_i(l_i, t) \ \boldsymbol{\psi}_i(l_i, t), \dot{\boldsymbol{\psi}}_i(l_i, t)] \tag{43}$$

We propose to obtain Lagrange's equations in terms of quasicoordinates by transforming Eqs. (37). To this end, we use the chain rule for derivatives with respect to vectors and consider Eq. (39) to obtain

$$\frac{\partial L}{\partial \dot{\boldsymbol{R}}_{o1}} = \frac{\partial (C_1\dot{\boldsymbol{R}}_{o1})^T}{\partial \dot{\boldsymbol{R}}_{o1}}\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} = C_1^T\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} \tag{44a}$$

$$\frac{\partial L}{\partial \boldsymbol{R}_{o1}} = \frac{\partial L^*}{\partial \boldsymbol{R}_{o1}} \tag{44b}$$

But, it is shown in the Appendix that the matrix of direction cosines C_i and quasi-velocity vector $\boldsymbol{\omega}_i$ satisfy the relation

$$\dot{C}_i = \tilde{\omega}_i^T C_i \tag{45}$$

so that differentiating Eq. (44a) with respect to time, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{R}}_{o1}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(C_1^T\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}}\right) = C_1^T\tilde{\omega}_1\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} + C_1^T\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}}\right) \tag{46}$$

Then, inserting Eqs. (44b) and (46) into Eq. (37a) and premultiplying by C_1, we obtain the translational Lagrange equations in terms of quasicoordinates

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}}\right) + \tilde{\omega}_1\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} - C_1\frac{\partial L^*}{\partial \boldsymbol{R}_{o1}} = \boldsymbol{F}_1^* \tag{47}$$

where

$$\boldsymbol{F}_1^* = C_1\boldsymbol{F}_1 \tag{48}$$

is the resultant force acting on body 1 in terms of body-axes components.

As far as the rotational motion is concerned, we consider first the equations for body 1. Using the chain rule for derivatives with respect to vectors once again and using Eq. (40), we obtain

$$\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}_1} = \frac{\partial (D_1\dot{\boldsymbol{\theta}}_1)^T}{\partial \dot{\boldsymbol{\theta}}_1}\frac{\partial L^*}{\partial \boldsymbol{\omega}_1} = D_1^T\frac{\partial L^*}{\partial \boldsymbol{\omega}_1} \tag{49a}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_1} = \frac{\partial L^*}{\partial \boldsymbol{\theta}_1} + \frac{\partial (C_1\dot{\boldsymbol{R}}_{o1})^T}{\partial \boldsymbol{\theta}_1}\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} + \frac{\partial (D_1\dot{\boldsymbol{\theta}}_1)^T}{\partial \boldsymbol{\theta}_1}\frac{\partial L^*}{\partial \boldsymbol{\omega}_1} \tag{49b}$$

Moreover, Eq. (A26) from the Appendix, with $\boldsymbol{a}$ replaced by $\dot{\boldsymbol{R}}_{o1}$ yields the relation

$$\frac{\partial (C_1\dot{\boldsymbol{R}}_{o1})^T}{\partial \boldsymbol{\theta}_1} = -D_1^T\tilde{V}_{o1} \tag{50}$$

and Eq. (A24) shows that

$$\dot{D}_1^T = \frac{\partial (D_1\dot{\boldsymbol{\theta}}_1)^T}{\partial \boldsymbol{\theta}_1} + D_1^T\tilde{\omega}_1 \tag{51}$$

Hence, using Eqs. (49–51), we can write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}_1}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}_1} &= \left[\dot{D}_1^T - \frac{\partial (D_1\dot{\boldsymbol{\theta}}_1)^T}{\partial \boldsymbol{\theta}_1}\right]\frac{\partial L^*}{\partial \boldsymbol{\omega}_1} \\
&\quad + D_1^T\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right) + D_1^T\tilde{V}_{o1}\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} - \frac{\partial L^*}{\partial \boldsymbol{\theta}_1} \\
&= D_1^T\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right) + \tilde{V}_{o1}\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} + \tilde{\omega}_1\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right] - \frac{\partial L^*}{\partial \boldsymbol{\theta}_1}
\end{aligned} \tag{52}$$

Inserting Eq. (52) into Eq. (37b) and premultiplying the result by D_1^{-T}, where the superscript $-T$ denotes the inverse of the transposed matrix, we obtain the rotational Lagrange equations for the first body in terms of quasicoordinates,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right) + \tilde{V}_{o1}\frac{\partial L^*}{\partial \boldsymbol{V}_{o1}} + \tilde{\omega}_1\frac{\partial L^*}{\partial \boldsymbol{\omega}_1} - D_1^{-T}\frac{\partial L^*}{\partial \boldsymbol{\theta}_1} = \boldsymbol{M}_1^* \tag{53}$$

where

$$M_1^* = D_1^{-T} M_1 \tag{54}$$

is the resultant torque acting on body 1 in terms of body-axes components. The equations of motion for the remaining bodies can be obtained in the same manner, except that V_{oi} $(i = 2, 3, \ldots, N)$ are not independent, as can be concluded from Eqs. (10). Hence, from Eq. (53), the remaining rotational Lagrange's equations in terms of quasicoordinates are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \omega_i}\right) + \tilde{\omega}_i \frac{\partial L^*}{\partial \omega_i} - D_i^{-T}\frac{\partial L^*}{\partial \theta_i} = M_i^* \quad i = 2, 3, \ldots, N \tag{55}$$

where

$$M_i^* = D_i^{-T} M_i \qquad i = 2, 3, \ldots, N \tag{56}$$

Equations (47), (53), and (55) can be cast in a single matrix equation. Indeed, recalling Eqs. (29), (38), (41b), and (42b), the rigid-body Lagrange equations of motion in terms of quasicoordinates can be written in the compact form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial w}\right) + H\frac{\partial L}{\partial w} - B^T\frac{\partial L}{\partial q} = Q^* \tag{57}$$

where the asterisk in L^* was dropped for convenience. Moreover,

$$H = \begin{bmatrix} \tilde{\omega}_1 & 0 & 0 & \cdots & 0 \\ \tilde{V}_{o1} & \tilde{\omega}_1 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{\omega}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{\omega}_N \end{bmatrix} \tag{58}$$

and

$$Q^* = B^T Q = \begin{bmatrix} F_1^{*T} & M_1^{*T} & M_2^{*T} & \cdots & M_N^{*T} \end{bmatrix}^T \tag{59}$$

The hybrid set of equations of motion is completed by adjoining to Eq. (57) the partial differential equations for the elastic motions [Eqs. (34b) and (34c)] and the associated boundary conditions [Eqs. (36)].

V. Explicit Hybrid Equations of Motion for Flexible Multibody Systems

Using Eqs. (16) and (17), we can write the kinetic energy in the form

$$\begin{aligned} T &= \sum_{i=1}^N \int_0^{l_i} \hat{T}_i \mathrm{d}x_i \\ &= \frac{1}{2}\sum_{i=1}^N \Bigg[m_i V_{oi}^T V_{oi} + \Omega_{ri}^T J_{ti} \Omega_{ri} + \int_0^{l_i} \rho_i \dot{u}_i^T \dot{u}_i \mathrm{d}x_i \\ &\quad + \int_0^{l_i} \dot{\psi}_i^T \hat{J}_{ci} \dot{\psi}_i \mathrm{d}x_i + 2V_{oi}^T\left(\tilde{S}_i^T \Omega_{ri} + \int_0^{l_i} \rho_i \dot{u}_i \mathrm{d}x_i\right) \\ &\quad + 2\Omega_{ri}^T \int_0^{l_i} (\tilde{\hat{S}}_i \dot{u}_i + \hat{J}_{ci}\dot{\psi}_i)\,\mathrm{d}x_i \Bigg] \end{aligned} \tag{60}$$

and we observe that T does not depend explicitly on the quasi-velocities V_{o1} and ω_i $(i = 1, 2, \ldots, N)$, but on V_{oi} and Ω_{ri} $(i = 1, 2, \ldots, N)$. To resolve this inconvenience, we make use of the discrete step function γ_i, defined by

$$\gamma_i = \begin{cases} 0 & \text{if } i = -1, -2, -3, \ldots \\ 1 & \text{if } i = 0, 1, 2, 3, \ldots \end{cases} \tag{61}$$

and then make repeated use of Eqs. (10) and (13) to establish the relations

$$\Omega_{ri} = \sum_{j=1}^N C_{ij}^* [\gamma_{i-j}\omega_j + \gamma_{i-j-1}\Omega_{ej}(l_j, t)] \tag{62a}$$

$$\begin{aligned} V_{oi} &= C_{i1}^* V_{o1} + \sum_{j=1}^N \gamma_{i-j-1} C_{ij}^*\left[\tilde{u}_{cj}^T \Omega_{rj} + v_j(l_j, t)\right] \\ &= C_{i1}^* V_{o1} + \sum_{j=1}^N \big[\Gamma_{ij}\omega_j + \Gamma_{i,j+1}\Omega_{ej}(l_j, t) \\ &\quad + \gamma_{i-j-1} C_{ij}^* v_j(l_j, t)\big] \end{aligned} \tag{62b}$$

$$\dot{\Omega}_{ri} = \sum_{j=1}^N C_{ij}^*[\gamma_{i-j}\dot{\omega}_j + \gamma_{i-j-1}\dot{\Omega}_{ej}(l_j, t)] + d_{\Omega i} \tag{63a}$$

$$\begin{aligned} \dot{V}_{oi} &= C_{i1}^* \dot{V}_{o1} + \sum_{j=1}^N [\Gamma_{ij}\dot{\omega}_j + \Gamma_{i,j+1}\dot{\Omega}_{ej}(l_j, t) \\ &\quad + \gamma_{i-j-1} C_{ij}^* \dot{v}_j(l_j, t)] + d_{Vi} \end{aligned} \tag{63b}$$

$$\delta\Theta_{ri}^* = \sum_{j=1}^N C_{ij}^*[\gamma_{i-j}\delta\theta_j^* + \gamma_{i-j-1}\delta\psi_j(l_j, t)] \tag{63c}$$

$$\begin{aligned} \delta R_{oi}^* &= C_{i1}^*\,\delta R_{o1}^* + \sum_{j=1}^N [\Gamma_{ij}\delta\theta_j^* + \Gamma_{i,j+1}\delta\psi_j(l_j, t) \\ &\quad + \gamma_{i-j-1} C_{ij}^*\,\delta u_j(l_j, t)] \end{aligned} \tag{63d}$$

in which C_{ij}^* is simply the matrix of direction cosines of axes $x_i y_i z_i$ with respect to axes $x_j y_j z_j$, defined for all indices i, j between 1 and N, and consequently

$$C_{ij}^* = \prod_{k=j+1}^{i} C_k^* \qquad 1 \le j < i \le N \tag{64a}$$

$$C_{ii}^* = I \qquad 1 \le i \le N \tag{64b}$$

$$\left(C_{ij}^*\right)^T = C_{ji}^* \qquad 1 \le i, j, k \le N \tag{64c}$$

$$C_{ik}^* C_{kj}^* = C_{ij}^* \qquad 1 \le i, j, k \le N \tag{64d}$$

The other quantities appearing explicitly or implicitly in Eqs. (62) and (63) are given by

$$u_{ci} = [l_i \quad u_{yi}(l_i, t) \quad u_{zi}(l_i, t)]^T \tag{65a}$$

$$\Gamma_{ij} = \sum_{k=j}^{i-1} C_{ik}^* \tilde{u}_{ck}^T C_{kj}^* \tag{65b}$$

$$d_{\Omega i} = \sum_{j=1}^N \dot{C}_{ij}^*[\gamma_{i-j}\omega_j + \gamma_{i-j-1}\Omega_{ej}(l_j, t)] \tag{65c}$$

$$\begin{aligned} d_{Vi} &= \dot{C}_{i1}^* V_{o1} + \sum_{j=1}^{i-1} \Big\{\dot{C}_{ij}^*\left[\tilde{u}_{cj}^T \Omega_{rj} + v_j(l_j, t)\right] \\ &\quad + C_{ij}^*\left[\tilde{\Omega}_{rj} v_j(l_j, t) + \tilde{u}_{cj}^T d_{\Omega j}\right]\Big\} \end{aligned} \tag{65d}$$

$$\dot{C}_{ij}^* = \widetilde{\left(-\Omega_{ri} + C_{ij}^*\Omega_{rj}\right)} C_{ij}^* \tag{65e}$$

We also note that C_{ij}^* depends only on $\boldsymbol{\theta}_k$ for $\min(i, j) < k \leq \max(i, j)$ and on $\boldsymbol{\psi}_k(l_k, t)$ for values of k satisfying $\min(i, j) \leq k < \max(i, j)$. Hence, using Eqs. (A26) and (A27), we can derive the relations

$$\frac{\partial (C_{ij}^* \boldsymbol{a})^T}{\partial \boldsymbol{\theta}_k} = (\gamma_{j-k} - \gamma_{i-k}) D_k^T C_{kj}^* \tilde{a} C_{ji}^* \tag{66a}$$

provided $\boldsymbol{a}$ does not depend on $\boldsymbol{\theta}_k$, and

$$\frac{\partial (C_{ij}^* \boldsymbol{a})^T}{\partial \boldsymbol{\psi}_k(l_k, t)} = (\gamma_{j-k-1} - \gamma_{i-k-1}) E_k(l_k) C_{kj}^* \tilde{a} C_{ji}^* \tag{66b}$$

provided $\boldsymbol{a}$ does not depend on $\boldsymbol{\psi}_k(l_k, t)$. Some other relations that will prove useful are as follows:

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{R}_{o1}} = 0 \tag{67a}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{\theta}_k} = D_k^T \sum_{j=1}^{N} (\gamma_{j-k} - \gamma_{i-k}) C_{kj}^* [\gamma_{i-j} \tilde{\omega}_j + \gamma_{i-j-1} \tilde{\Omega}_{ej}(l_j, t)] C_{ji}^* \tag{67b}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{u}_k(l_k, t)} = 0 \tag{67c}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{\psi}_k(l_k, t)} = E_k(l_k, t) \sum_{j=1}^{N} (\gamma_{j-k-1} - \gamma_{i-k-1}) \times C_{kj}^* [\gamma_{i-j} \tilde{\omega}_j + \gamma_{i-j-1} \tilde{\Omega}_{ej}(l_j, t)] C_{ji}^* \tag{67d}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{V}_{o1}} = 0 \tag{67e}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{\omega}_k} = \gamma_{i-k} C_{ik}^* \tag{67f}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{v}_k(l_k, t)} = 0 \tag{67g}$$

$$\frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{\Omega}_{ek}(l_k, t)} = \gamma_{i-k-1} C_{ik}^* \tag{67h}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{R}_{o1}} = 0 \tag{68a}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{\theta}_k} = D_k^T \Bigg[(\gamma_{1-k} - \gamma_{i-k}) C_{k1}^* \tilde{V}_{o1} C_{1i}^* + \sum_{j=1}^{i-1} \bigg\{ (\gamma_{j-k} - \gamma_{i-k}) C_{kj}^* \Big[\widetilde{\tilde{u}_{cj}^T \boldsymbol{\Omega}_{rj} + \boldsymbol{v}_j(l_j, t)} \Big] + D_k^{-T} \frac{\partial \boldsymbol{\Omega}_{rj}^T}{\partial \boldsymbol{\theta}_k} \tilde{u}_{cj} \bigg\} C_{ji}^* \Bigg] \tag{68b}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{u}_k(l_k, t)} = -\sum_{j=1}^{i-1} \tilde{\Omega}_{rj} C_{ji}^* \tag{68c}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{\psi}_k(l_k, t)} = E_k(l_k, t) \Bigg[-\gamma_{i-k-1} C_{k1}^* \tilde{V}_{o1} C_{1i}^* + \sum_{j=1}^{i-1} \bigg\{ (\gamma_{j-k-1} - \gamma_{i-k-1}) C_{kj}^* \Big[\widetilde{\tilde{u}_{cj}^T \boldsymbol{\Omega}_{rj} + \boldsymbol{v}_j(l_j, t)} \Big] + E_k^{-1}(l_k, t) \frac{\partial \boldsymbol{\Omega}_{rj}^T}{\partial \boldsymbol{\psi}_k(l_k, t)} \tilde{u}_{cj} \bigg\} C_{ji}^* \Bigg] \tag{68d}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{V}_{o1}} = C_{1i}^* \tag{68e}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{\omega}_k} = \Gamma_{ik}^T \tag{68f}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{v}_k(l_k, t)} = \gamma_{i-k-1} C_{ki}^* \tag{68g}$$

$$\frac{\partial \boldsymbol{V}_{oi}^T}{\partial \boldsymbol{\Omega}_{ek}(l_k, t)} = \Gamma_{i,k+1}^T \tag{68h}$$

Then, using the chain rule for vectors when needed, we obtain the momenta

$$\boldsymbol{p}_{Vo1} = \frac{\partial L}{\partial \boldsymbol{V}_{o1}} = \sum_{i=1}^{N} C_{1i}^* \frac{\partial L_i}{\partial \boldsymbol{V}_{oi}} \tag{69a}$$

$$\boldsymbol{p}_{\omega j} = \frac{\partial L}{\partial \boldsymbol{\omega}_j} = \sum_{i=1}^{N} \left(\Gamma_{ij}^T \frac{\partial L_i}{\partial \boldsymbol{V}_{oi}} + \gamma_{i-j} C_{ij}^* \frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}} \right) \tag{69b}$$

where

$$\frac{\partial L_i}{\partial \boldsymbol{V}_{oi}} = m_i \boldsymbol{V}_{oi} + \tilde{S}_i^T \boldsymbol{\Omega}_{ri} + \int_0^{l_i} \rho_i \dot{\boldsymbol{u}}_i \mathrm{d}x_i \tag{70a}$$

$$\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}} = J_{ti} \boldsymbol{\Omega}_{ri} + \tilde{S}_i \boldsymbol{V}_{oi} + \int_0^{l_i} (\hat{\tilde{S}}_i \dot{\boldsymbol{u}}_i + \hat{J}_{ci} \dot{\boldsymbol{\psi}}) \mathrm{d}x_i \tag{70b}$$

For future reference, we also indicate that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_i}{\partial \boldsymbol{V}_{oi}} \right) = m_i \dot{\boldsymbol{V}}_{oi} + \tilde{S}_i^T \dot{\boldsymbol{\Omega}}_{ri} + \int_0^{l_i} \rho_i \ddot{\boldsymbol{u}}_i \mathrm{d}x_i + \boldsymbol{d}_{tVi} \tag{71a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}} \right) = J_{ti} \dot{\boldsymbol{\Omega}}_{ri} + \tilde{S}_i \dot{\boldsymbol{V}}_{oi} + \int_0^{l_i} (\hat{\tilde{S}}_i \ddot{\boldsymbol{u}}_i + \hat{J}_{ci} \ddot{\boldsymbol{\psi}}_i) \mathrm{d}x_i + \boldsymbol{d}_{t\Omega i} \tag{71b}$$

where

$$\boldsymbol{d}_{tVi} = \tilde{\Omega}_{ri} \int_0^{l_i} \rho_i \dot{\boldsymbol{u}}_i \mathrm{d}x_i \tag{72a}$$

$$\boldsymbol{d}_{t\Omega i} = \dot{J}_{ti} \boldsymbol{\Omega}_{ri} - \tilde{V}_{oi} \int_0^{l_i} \rho_i \dot{\boldsymbol{u}}_i \mathrm{d}x_i \tag{72b}$$

$$\dot{J}_{ti} = \int_0^{l_i} \rho_i \left[\tilde{\dot{u}}_i (x_i \tilde{e}_1 + \tilde{u}_i)^T + (x_i \tilde{e}_1 + \tilde{u}_i) \tilde{\dot{u}}_i^T \right] \mathrm{d}x_i \tag{72c}$$

and

$$\dot{\boldsymbol{p}}_{Vo1} = \left(\sum_{i=1}^{N} m_i \right) \dot{\boldsymbol{V}}_{o1} + \sum_{j=1}^{N} \left[\sum_{i=1}^{N} \left(m_i C_{1i}^* \Gamma_{ij} + \gamma_{i-j} C_{1i}^* \tilde{S}_i^T C_{ij}^* \right) \right] \dot{\boldsymbol{\omega}}_j + \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \gamma_{i-j-1} m_i C_{1j}^* \right) \dot{\boldsymbol{v}}_j(l_j, t) + \sum_{j=1}^{N} \left[\sum_{i=1}^{N} \left(m_i C_{1i}^* \Gamma_{i,j+1} + \gamma_{i-j-1} C_{1i}^* \tilde{S}_i^T C_{ij}^* \right) \right] \dot{\boldsymbol{\Omega}}_{ej}(l_j, t) + \sum_{i=1}^{N} \left(C_{1i}^* \int_0^{l_i} \rho_i \ddot{\boldsymbol{u}}_i \mathrm{d}x_i \right) + \sum_{i=1}^{N} \left[\dot{C}_{1i}^* \frac{\partial L_i}{\partial \boldsymbol{V}_{oi}} + C_{1i}^* \left(m_i \boldsymbol{d}_{Vi} + \tilde{S}_i^T \boldsymbol{d}_{\Omega i} + \boldsymbol{d}_{tVi} \right) \right] \tag{73a}$$

$$\dot{p}_{\omega j} = \left[\sum_{i=1}^{N}\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)C_{i1}^*\right]\dot{V}_{o1}$$
$$+\sum_{k=1}^{N}\left\{\sum_{i=1}^{N}\left[\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)\Gamma_{ik}\right.\right.$$
$$\left.\left.+\gamma_{i-k}\left(\Gamma_{ij}^T\tilde{S}_i^T + \gamma_{i-j}C_{ij}^*J_{ti}\right)C_{ik}^*\right]\right\}\dot{\omega}_k$$
$$+\sum_{k=1}^{N}\left[\sum_{i=1}^{N}\gamma_{i-k-1}\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)C_{ik}^*\right]\dot{v}_k(l_k,t)$$
$$+\sum_{k=1}^{N}\left\{\sum_{i=1}^{N}\left[\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)\Gamma_{i,k+1}\right.\right.$$
$$\left.\left.+\gamma_{i-k-1}\left(\Gamma_{ij}^T\tilde{S}_i^T + \gamma_{i-j}C_{ij}^*J_{ti}\right)C_{ik}^*\right]\right\}\dot{\Omega}_{ek}(l_k,t)$$
$$+\sum_{i=1}^{N}\left[\int_0^{l_i}\left(\rho_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{\hat{S}}_i\right)\ddot{u}_i\,\mathrm{d}x_i\right]$$
$$+\sum_{i=1}^{N}\left(\gamma_{i-j}C_{ij}^*\int_0^{l_i}\hat{J}_{ci}\ddot{\psi}_i\,\mathrm{d}x_i\right)$$
$$+\sum_{i=1}^{N}\left[\dot{\Gamma}_{ij}^T\frac{\partial L_i}{\partial V_{oi}} + \gamma_{i-j}\dot{C}_{ij}^*\frac{\partial L_i}{\partial \Omega_{ri}} + \Gamma_{ij}^T d_{tVi}\right.$$
$$+\gamma_{i-j}C_{ij}^* d_{t\Omega i} + \left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)d_{Vi}$$
$$\left.+\left(\Gamma_{ij}^T\tilde{S}_i^T + \gamma_{i-j}C_{ij}^*J_{ti}\right)d_{\Omega i}\right] \tag{73b}$$

We also define equivalent forces and moments

$$F_{p1}^* = C_1\frac{\partial L}{\partial R_{o1}} = 0 \tag{74a}$$

$$M_{pj}^* = D_j^{-T}\frac{\partial L}{\partial \theta_j} = D_j^{-T}\sum_{i=1}^{N}\left(\frac{\partial V_{oi}^T}{\partial \theta_j}\frac{\partial L}{\partial V_{oi}} + \frac{\partial \Omega_{ri}^T}{\partial \theta_j}\frac{\partial L}{\partial \Omega_{ri}}\right) \tag{74b}$$

and the remaining pertinent terms

$$\frac{\partial L}{\partial u_j(l_j,t)} = \sum_{i=1}^{N}\frac{\partial V_{oi}^T}{\partial u_j(l_j,t)}\frac{\partial L_i}{\partial V_{oi}} \tag{75a}$$

$$\frac{\partial L}{\partial v_j(l_j,t)} = \sum_{i=1}^{N}\frac{\partial V_{oi}^T}{\partial v_j(l_j,t)}\frac{\partial L_i}{\partial V_{oi}} \tag{75b}$$

$$\frac{\partial L}{\partial \psi_j(l_j,t)} = \sum_{i=1}^{N}\left(\frac{\partial V_{oi}^T}{\partial \psi_j(l_j,t)}\frac{\partial L}{\partial V_{oi}} + \frac{\partial \Omega_{ri}^T}{\partial \psi_j(l_j,t)}\frac{\partial L}{\partial \Omega_{ri}}\right) \tag{75c}$$

$$\frac{\partial L}{\partial \Omega_{ej}(l_j,t)} = \sum_{i=1}^{N}\left(\frac{\partial V_{oi}^T}{\partial \Omega_{ej}(l_j,t)}\frac{\partial L}{\partial V_{oi}} + \frac{\partial \Omega_{ri}^T}{\partial \Omega_{ej}(l_j,t)}\frac{\partial L}{\partial \Omega_{ri}}\right) \tag{75d}$$

in which some of the partial derivatives are given by Eqs. (67).

Finally, adjoining the kinematic relations expressed by Eqs. (9), (11), (39), and (40) and inserting Eqs. (68–70) into Eqs. (34b), (34c), and (57), we obtain the hybrid state equations in terms of quasicoordinates:

$$\dot{R}_{o1} = C_1^T V_{o1} \tag{76a}$$

$$\dot{\theta}_i = D_i^{-1}\omega_i \qquad i = 1,2,\ldots,N \tag{76b}$$

$$\dot{u}_i(x_i,t) = v_i(x_i,t) \qquad i = 1,2,\ldots,N \tag{76c}$$

$$\dot{\psi}_i(x_i,t) = \Omega_{ei}(x_i,t) \qquad i = 1,2,\ldots,N \tag{76d}$$

$$\dot{p}_{Vo1} = -\tilde{\omega}_1 p_{Vo1} + F_1^* \tag{76e}$$

$$\dot{p}_{\omega 1} = -\tilde{V}_{o1}p_{Vo1} - \tilde{\omega}_1 p_{\omega 1} + M_{p1}^* + M_{o1}^* \tag{76f}$$

$$\dot{p}_{\omega i} = -\tilde{\omega}_i p_{\omega i} + M_{pi}^* + M_{oi}^* \qquad i = 2,3,\ldots,N \tag{76g}$$

$$\rho_i\left[\dot{v}_{yi} + \dot{V}_{oyi} + x_i\dot{\Omega}_{rzi} - u_{zi}\dot{\Omega}_{rxi} - 2\Omega_{rxi}v_{zi} + \Omega_{rzi}V_{oxi}\right.$$
$$\left.-\Omega_{rxi}V_{ozi} + x_i\Omega_{rxi}\Omega_{ryi} - \left(\Omega_{rxi}^2 + \Omega_{rzi}^2\right)u_{yi} + \Omega_{ryi}\Omega_{rzi}u_{zi}\right]$$
$$-[k_{yi}G_iA_i(u_{yi}' - \psi_{zi})]' = f_{yi} \tag{76h}$$

$$\rho_i\left[\dot{v}_{zi} + \dot{V}_{ozi} - x_i\dot{\Omega}_{ryi} + u_{yi}\dot{\Omega}_{rxi} + 2\Omega_{rxi}v_{yi}\right.$$
$$+\Omega_{rxi}V_{oyi} - \Omega_{ryi}V_{oxi} + x_i\Omega_{rxi}\Omega_{rzi}$$
$$\left.-\left(\Omega_{rxi}^2 + \Omega_{ryi}^2\right)u_{zi} + \Omega_{ryi}\Omega_{rzi}u_{yi}\right]$$
$$-[k_{zi}G_iA_i(u_{zi}' + \psi_{yi})]' = f_{zi} \tag{76i}$$

$$\hat{J}_{xixi}(\dot{\Omega}_{exi} + \dot{\Omega}_{rxi}) - (k_{xi}G_iI_{xi}\psi_{xi}')' = m_{xi} \tag{76j}$$

$$\hat{J}_{yiyi}(\dot{\Omega}_{eyi} + \dot{\Omega}_{ryi}) + k_{zi}G_iA_i(u_{zi}' + \psi_{yi}) - (E_iI_{yi}\psi_{yi}')' = m_{yi} \tag{76k}$$

$$\hat{J}_{zizi}(\dot{\Omega}_{ezi} + \dot{\Omega}_{rzi}) - k_{yi}G_iA_i(u_{yi}' - \psi_{zi}) - (E_iI_{zi}\psi_{zi}')' = m_{zi} \tag{76l}$$

The associated boundary conditions [Eqs. (36)] are given by

$$u_i(0,t) = \mathbf{0} \qquad i = 1,2,\ldots,N \tag{77a}$$

$$\psi_i(0,t) = \mathbf{0} \qquad i = 1,2,\ldots,N \tag{77b}$$

$$\left.\frac{\partial \hat{L}_i}{\partial u_i'}\right|_{x_i=l_i} - \left\{\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial v_i(l_i,t)}\right] - \frac{\partial L}{\partial u_i(l_i,t)}\right\} = \mathbf{U}_i \qquad i = 1,2,\ldots,N-1 \tag{77c}$$

$$\left.\frac{\partial \hat{L}_i}{\partial \psi_i'}\right|_{x_i=l_i} - \left\{\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \Omega_{e,i}(l_i,t)}\right] - \frac{\partial L}{\partial \psi_i(l_i,t)}\right\} = \Psi_i \qquad i = 1,2,\ldots,N-1 \tag{77d}$$

$$\left.\frac{\partial \hat{L}_N}{\partial u_N'}\right|_{x_N=l_N} = \mathbf{0} \tag{77e}$$

$$\left.\frac{\partial \hat{L}_N}{\partial \psi_N'}\right|_{x_N=l_N} = \mathbf{0} \tag{77f}$$

and the generalized forces and torques are given by

$$F_1^* = \sum_{i=1}^{N}C_{1i}^*F_{ri}^* \tag{78a}$$

$$M_1^* = \sum_{i=1}^{N}\left(\Gamma_{i1}^TF_{ri}^* + C_{1i}^*M_{ri}^*\right) \tag{78b}$$

$$M_i^* = M_{oi}^* + \sum_{j=1}^{N}\left(\Gamma_{ji}^TF_{rj}^* + \gamma_{j-i}C_{ij}^*M_{rj}^*\right) \qquad i = 2,3,\ldots,N \tag{78c}$$

$$U_i = \sum_{j=1}^{N}\gamma_{j-i-1}C_{ij}^*F_{rj}^* \qquad i = 1,2,\ldots,N-1 \tag{78d}$$

$$\Psi_i = \sum_{j=1}^{N} \left(\Gamma_{j,i+1}^T F_{rj}^* + \gamma_{j-i-1} C_{ij}^* M_{rj}^*\right)$$

$$i = 1, 2, \ldots, N-1 \qquad (78e)$$

where we have made use of Eqs. (27), (32a), (63c), and (63d).

VI. Summary and Conclusions

In recent years, there has been an increasing interest in deriving the equations of motion for flexible multibody systems by treating the mass and stiffness of the bodies as distributed parameters. The equations of motion are generally derived by means of the extended Hamilton principle, leading to a hybrid set of equations, where hybrid is to be taken in the sense that the rigid-body translations and rotations of the bodies are described by ordinary differential equations and the elastic motions are described by partial differential equations with appropriate boundary conditions. In earlier investigations, the rigid-body rotations were described by Eulerian-type angles, which tend to complicate unduly the equations of motion, unless the motion remains planar.

This paper presents a mathematical formulation for flexible multibodies in terms of quasicoordinates, which permits the derivation of the equations for general rigid-body motions with considerably more ease than by using Eulerian-type angles. As an added feature, the equations for the elastic motions include rotatory inertia and shear deformation effects. The equations of motion are cast in state form, making them suitable for control design.

Appendix: Matrix Derivatives

Derivative Rules

If $A = [A_{ij}]$ is an $m \times n$ matrix, then we define the partial derivative of A with respect to a scalar τ to be the $m \times n$ matrix $\partial A/\partial \tau = [\partial A_{ij}/\partial \tau]$. If A is a function of time t, then the derivative of A with respect to t is denoted by $\dot{A} = \mathrm{d}A/\mathrm{d}t = [\mathrm{d}A_{ij}/\mathrm{d}t]$. Let $B = [B_{ij}]$ be an $M \times N$ matrix. Then, the derivative of a matrix with respect to a matrix, $\partial A/\partial B$, is the $mM \times nN$ matrix defined by

$$\frac{\partial A}{\partial B} = \begin{bmatrix} \dfrac{\partial A}{\partial B_{11}} & \dfrac{\partial A}{\partial B_{12}} & \cdots & \dfrac{\partial A}{\partial B_{1N}} \\ \dfrac{\partial A}{\partial B_{21}} & \dfrac{\partial A}{\partial B_{22}} & \cdots & \dfrac{\partial A}{\partial B_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial A}{\partial B_{M1}} & \dfrac{\partial A}{\partial B_{M2}} & \cdots & \dfrac{\partial A}{\partial B_{MN}} \end{bmatrix} \qquad (A1)$$

Furthermore, let L be a scalar and $f = [f_1 \cdots f_m]^T$, $q = [q_1 \cdots q_n]^T$, $z = [z_1 \cdots z_r]^T$ be column matrices. Then $\partial L/\partial q$ is a column matrix, $\partial f^T/\partial q$ is an $n \times m$ matrix, and $\partial f/\partial q^T = (\partial f^T/\partial q)^T$. The chain rules for differentiation have the form

$$\frac{\partial f^T}{\partial z} = \frac{\partial q^T}{\partial z}\frac{\partial f^T}{\partial q} \quad \text{or} \quad \frac{\partial f}{\partial z^T} = \frac{\partial f}{\partial q^T}\frac{\partial q}{\partial z^T} \qquad (A2)$$

$$\frac{\partial L}{\partial z} = \frac{\partial q^T}{\partial z}\frac{\partial L}{\partial q} \quad \text{or} \quad \frac{\partial L}{\partial z^T} = \frac{\partial L}{\partial q^T}\frac{\partial q}{\partial z^T} \qquad (A3)$$

Moreover,

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial q^T}\dot{q} \qquad (A4)$$

$$\frac{\partial (Aq)}{\partial q^T} = A \quad \text{or} \quad \frac{\partial (Aq)^T}{\partial q} = A^T \qquad (A5)$$

$$\frac{\partial\left(\frac{1}{2}q^T A q\right)}{\partial q} = Aq \qquad (A6)$$

provided A does not depend on q.

Proper Orthogonal Matrices

Throughout this paper, we encounter proper orthogonal matrices C, which are functions of three independent coordinates $\theta = [\theta_1\, \theta_2\, \theta_3]^T$. These matrices can be identified as matrices of direction cosines of one coordinate system $\xi_1\xi_2\xi_3$, with corresponding unit vectors b_1, b_2, b_3, with respect to another coordinate system $x_1x_2x_3$, with corresponding unit vectors n_1, n_2, n_3. Hence, letting $C = [C_{ij}]$, the entries C_{ij} can be expressed as

$$C_{ij} = b_i \cdot n_j \qquad i, j = 1, 2, 3 \qquad (A7)$$

which implies that

$$n_j = \sum_{k=1}^{3} (n_j \cdot b_k) b_k = \sum_{k=1}^{3} C_{kj} b_k \qquad j = 1, 2, 3 \qquad (A8)$$

At this point we wish to establish a relation between the body axes components of the angular velocity ω of coordinate system $\xi_1\xi_2\xi_3$ with respect to coordinate system $x_1x_2x_3$ and the time derivative of C_{ij} with respect to coordinate system $x_1x_2x_3$. First, recall[32] that ω is uniquely characterized by

$$\dot{b}_i = \omega \times b_i \qquad i = 1, 2, 3 \qquad (A9)$$

where in this case the "dot" requires holding n_1, n_2, n_3 constant. Then, taking the time derivative of Eq. (A7) and using Eqs. (A8) and (A9), and some identity involving scalar and vector products, we obtain

$$\dot{C}_{ij} = \dot{b}_i \cdot n_j = (\omega \times b_i) \cdot n_j = (b_i \times n_j) \cdot \omega$$

$$= \left(b_i \times \sum_{k=1}^{3} C_{kj} b_k\right) \cdot \omega = \sum_{k=1}^{3} C_{kj} (b_i \times b_k) \cdot \omega \qquad (A10)$$

Now we observe that $(b_i \times b_k) \cdot \omega$, where $i, k = 1, 2, 3$ are merely the entries of the 3×3 matrix

$$[\omega_{ik}] = \begin{bmatrix} 0 & b_3 \cdot \omega & -b_2 \cdot \omega \\ -b_3 \cdot \omega & 0 & b_1 \cdot \omega \\ b_2 \cdot \omega & -b_1 \cdot \omega & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} = \tilde{\omega}^T \qquad (A11)$$

where $[\omega_1\ \omega_2\ \omega_3]^T$ are the $\xi_1\xi_2\xi_3$ components of ω, and we have used the fact that b_1, b_2, b_3 form a right-handed set of unit vectors. Inserting Eqs. (A11) into Eq. (A10), we obtain

$$\dot{C}_{ij} = \sum_{k=1}^{3} \omega_{ik} C_{kj} \qquad (A12)$$

which can be expressed in the matrix form

$$\dot{C} = \tilde{\omega}^T C \qquad (A13)$$

The relationship between ω and $\dot{\theta}$ has the form

$$\omega = D(\theta)\dot{\theta} \qquad (A14)$$

We now propose to derive some relation between D and C. In the first place, taking the partial derivative of $CC^T = I$ with respect to θ_i, we obtain

$$C\frac{\partial C^T}{\partial \theta_i} + \frac{\partial C}{\partial \theta_i}C^T = C\frac{\partial C^T}{\partial \theta_i} + \left(C\frac{\partial C^T}{\partial \theta_i}\right)^T = 0 \qquad i = 1, 2, 3 \qquad (A15)$$

from which we conclude that the 3×3 matrix $C(\partial C^T/\partial \theta_i)$ is skew symmetric. We denote the matrix by

$$\tilde{S}_i = C\frac{\partial C^T}{\partial \theta_i} \qquad i = 1, 2, 3 \qquad (A16)$$

where $\tilde{S}_i$ is obtained from the column matrix $\boldsymbol{S}_i = [S_{1i}\ \ S_{2i}\ \ S_{3i}]^T$ in the usual manner. We now calculate the time derivative of C^T in the form

$$\dot{C}^T = \sum_{i=1}^{3} \frac{\partial C^T}{\partial \theta_i}\dot{\theta}_i = C^T \sum_{i=1}^{3}\left(C\frac{\partial C^T}{\partial \theta_i}\right)\dot{\theta}_i = C^T \sum_{i=1}^{3} \tilde{S}_i\dot{\theta}_i$$

$$= C^T \widetilde{\left(\sum_{i=1}^{3} \boldsymbol{S}_i\dot{\theta}_i\right)} = C^T(\widetilde{[\boldsymbol{S}_1\ \boldsymbol{S}_2\ \boldsymbol{S}_3]\dot{\boldsymbol{\theta}}}) = C^T(\widetilde{S\dot{\boldsymbol{\theta}}}) \qquad \text{(A17)}$$

Comparing Eqs. (A13), (A14), and (A16), we conclude that

$$S = [\boldsymbol{S}_1\quad \boldsymbol{S}_2\quad \boldsymbol{S}_3] = D \qquad \text{(A18)}$$

Equation (A17) relates C and D in an implicit manner.

Next, we wish to derive an expression for $\dot{D}$. Taking the partial derivative of Eq. (A16) with respect to θ_j and replacing $\tilde{S}_i$ by $\tilde{D}_i$, we obtain

$$\frac{\partial \tilde{D}_i}{\partial \theta_j} = \frac{\partial C}{\partial \theta_j}\frac{\partial C^T}{\partial \theta_i} + C\frac{\partial^2 C^T}{\partial \theta_j\,\partial \theta_i} = \left(C\frac{\partial C^T}{\partial \theta_j}\right)^T\left(C\frac{\partial C^T}{\partial \theta_i}\right)$$

$$+ C\frac{\partial^2 C^T}{\partial \theta_j\,\partial \theta_i} = \tilde{D}_j^T\tilde{D}_i + C\frac{\partial^2 C^T}{\partial \theta_j\,\partial \theta_i} = -\tilde{D}_j\tilde{D}_i + C\frac{\partial^2 C^T}{\partial \theta_j\,\partial \theta_i} \qquad \text{(A19)}$$

Interchanging i and j in Eq. (A19), we have

$$\frac{\partial \tilde{D}_j}{\partial \theta_i} = -\tilde{D}_i\tilde{D}_j + C\frac{\partial^2 C^T}{\partial \theta_i\,\partial \theta_j} \qquad \text{(A20)}$$

Then, subtracting Eq. (A20) from Eq. (A19), we can write

$$\frac{\partial \tilde{D}_i}{\partial \theta_j} - \frac{\partial \tilde{D}_j}{\partial \theta_i} = \tilde{D}_i\tilde{D}_j - \tilde{D}_j\tilde{D}_i = (\widetilde{\tilde{D}_i\boldsymbol{D}_j}) \qquad \text{(A21)}$$

which implies that

$$\frac{\partial \boldsymbol{D}_i}{\partial \theta_j} - \frac{\partial \boldsymbol{D}_j}{\partial \theta_i} = \tilde{D}_i\boldsymbol{D}_j \qquad \text{(A22)}$$

This formula can be used in turn to derive an expression for $\dot{D}$. First, we recall Eq. (A14) and write

$$\dot{\boldsymbol{D}}_i = \sum_{j=1}^{3}\frac{\partial \boldsymbol{D}_i}{\partial \theta_j}\dot{\theta}_j = \sum_{j=1}^{3}\left(\frac{\partial \boldsymbol{D}_j}{\partial \theta_i}\dot{\theta}_j + \tilde{D}\boldsymbol{D}_i\dot{\theta}_j\right)$$

$$= \frac{\partial\left(\sum_{j=1}^{3}\boldsymbol{D}_j\dot{\theta}_j\right)}{\partial \theta_i} + \tilde{D}_i\left(\sum_{j=1}^{3}\boldsymbol{D}_j\dot{\theta}_j\right)$$

$$= \frac{\partial(D\dot{\theta})}{\partial \theta_i} + \tilde{D}_i\omega = \frac{\partial(D\dot{\theta})}{\partial \theta_i} + \tilde{\omega}^T\boldsymbol{D}_i \qquad \text{(A23)}$$

This implies that

$$\dot{D}^T = \begin{bmatrix}\dot{\boldsymbol{D}}_1^T\\ \dot{\boldsymbol{D}}_2^T\\ \dot{\boldsymbol{D}}_3^T\end{bmatrix} = \frac{\partial(D\dot{\boldsymbol{\theta}})^T}{\partial \boldsymbol{\theta}} + D^T\tilde{\omega} \qquad \text{(A24)}$$

Next, we consider the partial derivative of $(C\boldsymbol{a})^T$ with respect to $\boldsymbol{\theta}$, where $\boldsymbol{a}$ does not depend on $\boldsymbol{\theta}$. First, we recall Eqs. (A16) and (A18) and write

$$\frac{\partial (C\boldsymbol{a})^T}{\partial \theta_i} = \boldsymbol{a}^T\frac{\partial C^T}{\partial \theta_i} = \boldsymbol{a}^T C^T\left(C\frac{\partial C^T}{\partial \theta_i}\right) = (C\boldsymbol{a})^T\tilde{D}_i$$

$$= -(C\boldsymbol{a})^T\tilde{D}_i^T = -(\tilde{D}_iC\boldsymbol{a})^T = [(\widetilde{C\boldsymbol{a}})\boldsymbol{D}_i]^T = \boldsymbol{D}_i^T(\widetilde{C\boldsymbol{a}})^T \qquad \text{(A25)}$$

which implies that

$$\frac{\partial (C\boldsymbol{a})^T}{\partial \theta} = \begin{bmatrix}-\boldsymbol{D}_1^T(\widetilde{C\boldsymbol{a}})\\ -\boldsymbol{D}_2^T(\widetilde{C\boldsymbol{a}})\\ -\boldsymbol{D}_3^T(\widetilde{C\boldsymbol{a}})\end{bmatrix} = -D^T(\widetilde{C\boldsymbol{a}}) \qquad \text{(A26)}$$

The companion formula

$$\frac{\partial (C^T\boldsymbol{a})^T}{\partial \theta} = D^T\tilde{a}C \qquad \text{(A27)}$$

can be derived in a similar manner.

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